

Estimates for Kantorovich functionals between solutions to Fokker – Planck – Kolmogorov equations with dissipative drifts

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Abstract

We obtain estimates for the Kantorovich functionals between solutions to different Fokker – Planck – Kolmogorov equations for measures with same diffusion part but different drifts and different initial conditions. We show possible applications of such estimates to the study of the well-posedness for nonlinear equations.

KEYWORDS: Fokker – Planck – Kolmogorov equation; Kantorovich distance; Nonlinear Fokker – Planck equation; Dissipative operator.

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1 Introduction.

In the present paper we derive and study estimates for the Kantorovich functionals between probability solutions for the linear Fokker – Planck – Kolmogorov (FPK) equations for probability measures μ_t and σ_t on \mathbb{R}^d , $t \in [0, T]$, with different drifts and different initial conditions

$$\begin{aligned}\partial_t \mu_t &= \text{trace}(Q(x, t) D^2 \mu_t) - \text{div}(B_1(x, t) \mu_t), & \mu|_{t=0} &= \mu_0 \\ \partial_t \sigma_t &= \text{trace}(Q(x, t) D^2 \sigma_t) - \text{div}(B_2(x, t) \sigma_t), & \sigma|_{t=0} &= \sigma_0.\end{aligned}$$

We also show an alternative method to the study of well-posedness and stability of solutions to the nonlinear FPK equations

$$\partial_t \rho_t = \text{trace}(Q(x, t) D^2 \rho_t) - \text{div}(B(\rho, x, t) \rho_t), \quad \rho|_{t=0} = \rho_0, \quad (1)$$

based on such estimates for linear equations.

Recently, FPK equations have been actively studied from the functional-analytical, variational and as well from the probabilistic point of view. Interesting connections between approaches have been found (a survey of the current state of studies is provided in [3]). Estimates connecting distances between solutions with distances between initial data and even coefficients play a great role not only for the study of such qualitative properties of solutions as uniqueness or stability, but also for numerical simulations. In this context estimates for distances between solutions to equations with different drift terms are particularly interesting.

In Section 1 we derive estimates for the Kantorovich functionals between solutions of FPK equations with different dissipative drifts. To do this, we partially use ideas from [13]. Since these ideas can not be

directly applied neither in the case of different drifts nor to nonlinear equations, new methods and ideas should be used. Extension to these cases has been done for the Kantorovich functionals with bounded cost functions. Moreover, we admit time-dependent coefficients and a non-unit diffusion matrix Q . We note that the requirement of dissipativity is not really restrictive – in most physical examples, the drift term is a minus gradient of a convex function, i.e. dissipative. Section 2 is concerned with applications of these estimates to the study of the well-posedness of the Cauchy problem for the nonlinear FPK equation. Well-posedness for the nonlinear equations has been studied by many authors even in a more general setting (see, for example, [7, 11, 9, 10]). However we present an alternative approach to this problem that is applicable in case of dissipative drifts. A similar method of treating well-posedness via estimates for the distances between solutions to linear equations was used in [5].

Let us introduce some notation and give basic definitions. By $C_0^\infty(\mathbb{R}^d)$ and $C_0^\infty(\mathbb{R}^d \times (0, T))$ we denote classes of infinitely smooth compactly supported functions on \mathbb{R}^d and $\mathbb{R}^d \times (0, T)$ respectively. For shortness of notation we shall always drop the subscript \mathbb{R}^d when integrating over the whole space. We shall say that a measure ρ on $\mathbb{R}^d \times [0, T]$ is given by a family of probability measures $(\rho_t)_{t \in [0, T]}$ on \mathbb{R}^d (and write $\rho(dx dt) = \rho_t(dx) dt$ or simply $\rho = \rho_t dt$), if $\rho_t \geq 0$, $\rho_t(\mathbb{R}^d) = 1$, for each Borel set U the function $t \mapsto \rho_t(U)$ is measurable and

$$\int_0^T \int \phi d\rho = \int_0^T \int \phi d\rho_t dt \quad \forall \phi \in C_0^\infty(\mathbb{R}^d \times (0, T)).$$

Given a probability measure ρ_0 on \mathbb{R}^d , a symmetric Borel matrix $Q(x, t)$ and a Borel mapping $B(x, t) : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$, consider the following Cauchy problem for the linear FPK equation

$$\partial_t \rho_t = \text{trace}(Q(x, t) D^2 \rho_t) - \text{div}(B(x, t) \rho_t), \quad \rho|_{t=0} = \rho_0. \quad (2)$$

Here D^2 denotes the Hessian matrix with respect to the spacial variables. Denote the elements of the diffusion matrix $Q(x, t)$ by $q^{ij}(x, t)$, $1 \leq i, j \leq d$ and the elements of the vector drift $B(x, t)$ by $b^j(x, t)$, $1 \leq j \leq d$. Set

$$L\phi = q^{ij}(x, t) \partial_{x_i x_j}^2 \phi + b^i(x, t) \partial_{x_i} \phi,$$

where summation over all repeated indices is taken. We shall say that a measure $\rho(dx dt) = \rho_t(dx) dt$ is a solution to the Cauchy problem (2), if the mappings $q^{ij}(x, t)$, $b^i(x, t)$, $1 \leq i, j \leq d$, are Borel and belong to $L^1(\rho, U \times [0, T])$ for each ball $U \subset \mathbb{R}^d$, and for each test function $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int \varphi d\rho_t = \int \varphi d\rho_0 + \int_0^t \int L\varphi d\rho_s ds \quad (3)$$

for all $t \in [0, T]$. Sometimes it is more convenient to use an equivalent definition (see [4]), more precisely, the identity

$$\int \phi(x, t) d\rho_t = \int \phi(x, 0) d\rho_0 + \int_0^t \int [\partial_s \phi + L\phi] d\rho_s ds, \quad (4)$$

for all $t \in [0, T]$ and all test functions $\psi \in C^{2,1}(\mathbb{R}^d \times [0, T]) \cap C(\mathbb{R}^d \times [0, T])$ that are identically zero outside some ball $U \subset \mathbb{R}^d$. If we know a priori that the drift term B is integrable over $\mathbb{R}^d \times [0, T]$ with respect to the measure $d\rho_s ds$ and ϕ is supported on the whole \mathbb{R}^d , but has two continuous bounded derivatives, then (4) also holds true for such ϕ (to show this, it suffices to use a standard truncation argument).

2 Estimates for the Kantorovich functionals between solutions to linear equations with different drifts

In this section, we shall focus on two solutions of the linear FPK equation with different initial conditions and different drifts. Fix $T > 0$. Given probability measures μ_0 and σ_0 on \mathbb{R}^d , a symmetric Borel matrix $Q(x, t)$ and Borel mappings $B_\mu, B_\sigma : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$, consider two Cauchy problems

$$\begin{aligned} \partial_t \mu_t &= \text{trace}(Q(x, t) D^2 \mu_t) - \text{div}(B_\mu(x, t) \mu_t), & \mu|_{t=0} &= \mu_0 \\ \partial_t \sigma_t &= \text{trace}(Q(x, t) D^2 \sigma_t) - \text{div}(B_\sigma(x, t) \sigma_t), & \sigma|_{t=0} &= \sigma_0. \end{aligned} \quad (5)$$

We emphasize that the indices μ and σ in the drift coefficients are merely used to distinguish the different drifts (by marking corresponding solutions), and it is not necessary to define B as a map on a space of measures.

Given a monotone nonnegative continuous function h on \mathbb{R} with $h(0) = 0$, introduce the Kantorovich h -cost functional between the probability measures μ and σ by

$$C_h(\mu, \sigma) := \inf_{\pi \in \Pi(\mu, \sigma)} \int_{\mathbb{R}^d \times \mathbb{R}^d} h(|x - y|) d\pi(x, y), \quad (6)$$

where $\Pi(\mu, \sigma)$ is the set of couplings between μ and σ . Recall that a probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ belongs to $\Pi(\mu, \sigma)$ iff $\pi(E \times \mathbb{R}^d) = \mu(E)$, $\pi(\mathbb{R}^d \times E) = \sigma(E)$ for each Borel set $E \subset \mathbb{R}^d$. If h is a concave function with $h(r) > 0$ for $r > 0$, then C_h defines a distance on the space of probability measures and turns it into a complete metric space with topology that coincides with the usual weak one (see [1, Proposition 7.1.5]). Another important example is given by $h(r) = \min\{r^p, 1\}$ for some $p \geq 1$. In this case $C_h^{1/p}$ turns the space of probability measures into a complete metric space. Moreover, convergence with respect to this metric is equivalent to the weak convergence (see [6, Th. 1.1.9]).

Further we assume that a monotone non-decreasing continuous bounded cost function h with $h(0) = 0$ is fixed. Set $\|h\|_\infty := \sup_{z \in \mathbb{R}^d} h(|z|) < \infty$.

Throughout the paper we assume that the following regularity condition holds:

(A1) The diffusion matrix $Q(x, t)$ has uniformly bounded elements with uniformly bounded first derivatives. Moreover, it is strictly elliptic: there exists $\nu > 0$ such that $\forall (x, t) \in \mathbb{R}^d \times [0, T]$

$$\langle Q(x, t)y, y \rangle \geq \nu|y|^2 \quad \forall y \in \mathbb{R}^d. \quad (7)$$

Theorem 2.1. *Let (A1) hold. Let $(\mu_t)_{t \in [0, T]}$ and $(\sigma_t)_{t \in [0, T]}$ be solutions to (5) with initial conditions μ_0 and σ_0 respectively. Suppose that the drift term B_μ is λ -dissipative in x , i.e.*

$$\langle B_\mu(x, t) - B_\mu(y, t), x - y \rangle \leq \lambda \|x - y\|^2 \quad (8)$$

for all $x, y \in \mathbb{R}^d$ and all $t \in [0, T]$. Let

$$B_\mu(x, t) - \lambda x, B_\sigma(x, t) - \lambda x \in L^2(\mathbb{R}^d \times [0, T], d(\mu_s + \sigma_s)ds) \quad (9)$$

Then

$$C_{h_{\lambda t}}(\mu_t, \sigma_t) \leq C_h(\mu_0, \sigma_0) + \|h\|_\infty \cdot \sqrt{\int_0^t \int \nu^{-1} |B_\mu - B_\sigma|^2 d\sigma_s ds} \sqrt{1 + \int_0^t \int \nu^{-1} |B_\mu - B_\sigma|^2 d\sigma_s ds}, \quad (10)$$

for all $t \in [0, T]$, where $h_s(r) := h(re^{-s})$.

Remark 2.1. *The bound (10) is obviously asymmetric in measure: we impose dissipativity on B_μ , and the integration in the right-hand side is taken over σ . This property might be interesting from the point of view of possible numerical simulations. Indeed, if we want to solve a FPK equation*

$$\partial_t \mu_t = \text{trace}(Q(x, t) D^2 \mu_t) - \text{div}(B(x, t) \mu_t)$$

with a dissipative drift B , we can approximate the drift with “better” drifts B_n and solve FPK equations with those drifts. Then (10) controls the distance between the desired solution μ_t and the approximative solution μ_t^n in terms of the distance between drifts integrated over the **known** solution μ^n .

Proof. Let $(\mu_t)_{t \in [0, T]}$ and $(\sigma_t)_{t \in [0, T]}$ satisfy the assumptions of the theorem. By virtue of [2] measures μ_t and σ_t have strictly positive densities with respect to Lebesgue measure on \mathbb{R}^d for each $t \in [0, T]$. We split the proof of (10) into several steps.

Step 1. Reduction to the dissipative case ($\lambda = 0$). We rescale the problem, keeping the cost function unchanged, in order to reduce the problem to the case of a dissipative drift B_μ . To this aim we use the rescaling procedure from [13] with the opposite sign (since our drift term and the drift term in the cited work have the opposite signs). For completeness, we provide the rescaling procedure: for $\lambda \neq 0$ define the change of time

$$s(t) := \int_0^t e^{-2\lambda r} dr = \frac{1 - e^{-2\lambda t}}{2\lambda}, \quad t(s) = \frac{-\ln(1 - 2\lambda s)}{2\lambda}, \quad s \in [0, S_\infty),$$

where $S_\infty = +\infty$ for $\lambda < 0$ and $S_\infty = 1/(2\lambda)$ for $\lambda > 0$. For measures μ_t and σ_t introduce their rescaled versions ρ_s^μ and ρ_s^σ : for each Borel set $E \subset \mathbb{R}^d$ define $\rho_s^w(E) := w_{t(s)}(e^{\lambda t(s)} E)$ for $w = \mu, \sigma$. We notice that $C_h(\rho_s^\mu, \rho_s^\sigma) = C_{h_{\lambda t}}(\mu_t, \sigma_t)$. Since B_μ is λ -dissipative, $A_\mu := B_\mu - \lambda I$ is dissipative. Define the rescaled diffusion coefficient by

$$\tilde{Q}(y, s) := Q(t(s), e^{\lambda t(s)} y)$$

and the rescaled drifts by

$$\tilde{B}_w(y, s) := e^{\lambda t(s)} B_w(t(s), e^{\lambda t(s)} y), \quad \tilde{A}_w(y, s) := e^{\lambda t(s)} A_w(t(s), e^{\lambda t(s)} y), \quad w = \mu, \sigma.$$

Note that \tilde{A}_μ is also a dissipative operator. The measure $\mu = \mu_t dt$ is a solution to

$$\partial_t \mu_t = \text{trace}(Q D^2 \mu_t) - \text{div}(B_\mu \mu_t)$$

iff the rescaled version $\rho^\mu = \rho_t^\mu dt$ is a solution to

$$\partial_t \rho_t^\mu = \text{trace}(\tilde{Q} D^2 \rho_t^\mu) - \text{div}(\tilde{A}_\mu \rho_t^\mu); \quad (11)$$

moreover, (9) holds true iff for all nonnegative $s_1 < s_2 \leq S(T) < S_\infty$ one has

$$\int_{s_1}^{s_2} \int |\tilde{A}_\mu(x, s)|^2 d\rho_s^\mu ds < +\infty.$$

The integrability statement follows immediately from the change of variables formula, the identity (11) can be checked explicitly: it suffices to consider the change of variables $X(x, t) := (e^{-\lambda t} x, s(t))$ and calculate the derivatives. Similar statement holds for σ and ρ^σ . This means that it is sufficient to prove (10) only in the case $\lambda = 0$, i.e. in the case of a dissipative drift term B_μ .

Step 2. Approximation of the drift term. We construct a family of smooth (in both variables) bounded Lipschitz (as functions of x) dissipative operators $A_k^\varepsilon(x, t)$, approximating the dissipative drift term $B_\mu(x, t)$.

For each t the operator $B_\mu(\cdot, t)$ can be approximated by Lipschitz in x bounded dissipative operators $A_k(\cdot, t)$ with bounded first order derivatives with respect to the spacial variables (see [13, Th. 2.4, 2.5]): for each $t \in [0, T]$

$$\lim_{k \rightarrow \infty} A_k(x, t) = B_\mu(x, t) \text{ for a.e. } (x) \in \mathbb{R}^d \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} |A_k(x, t)| \leq k + 1. \quad (12)$$

Fix some non-negative function $\eta \in C_0^\infty([0, T])$ such that $\|\eta\|_{L^1([0, T])} = 1$. Introduce a family of mollifiers $\eta_\varepsilon(t) := \eta(t/\varepsilon)$. Since for each k the mapping $A_k(x, t)$ is bounded, the mappings $A_k^\varepsilon(x, t) := \eta_\varepsilon(t) * A_k(x, t)$ have bounded derivatives of all orders with respect to t and converge to $A_k(x, t)$ as $\varepsilon \rightarrow 0$ for a.e. $(x, t) \in \mathbb{R}^d \times [0, T]$. Notice that A_k^ε also have bounded first order derivatives with respect to the spacial variables. Moreover, A_k^ε are dissipative in x :

$$\langle A_k^\varepsilon(x, t) - A_k^\varepsilon(y, t), x - y \rangle = \eta_\varepsilon(t) * \langle A_k(x, t) - A_k(y, t), x - y \rangle \leq 0.$$

Finally, we define operators $\mathcal{L}_k^\varepsilon$ as follows: for $(x, s) \in \mathbb{R}^d \times [0, T]$

$$\mathcal{L}_k^\varepsilon[\phi](x, s) := \text{trace}(Q(x, s) D^2 \phi(x, s)) + \langle A_k^\varepsilon(x, s), \nabla_x \phi(x, s) \rangle, \quad \phi(\cdot, s) \in C^2(\mathbb{R}^d).$$

Step 3. Reduction of the class of test functions. It is well-known (for example, [15, Th. 1.3]) that the problem (6) admits a dual formulation: define the class Φ_h as

$$\Phi_h := \{(\phi, \psi) \in L^1(\mu) \times L^1(\nu) : \phi(x) + \psi(y) \leq h(|x - y|)\}.$$

Hence

$$C_h(\mu, \sigma) = \sup_{(\phi, \psi) \in \Phi_h} \int \phi d\mu + \int \psi d\sigma. \quad (13)$$

An important observation ([16, Lemma 2.3]) is that in the case of a bounded cost function h the supremum in the dual problem (13) may be taken over a smaller class of functions Φ_h^δ for any $\delta > 0$, where

$$\Phi_h^\delta := \Phi_h \cap C_0^\infty(\mathbb{R}^d) \cap \{(\phi, \psi) : \inf \psi > -\delta \text{ and } \sup \psi \leq \|h\|_\infty\}. \quad (14)$$

The proof is based on the fact that functions φ and ψ can be shifted by different constants and truncated in such way that the new pair (φ_0, ψ_0) is still admissible and the value in (13) doesn't decrease:

$$\int \varphi_0 d\mu + \int \psi_0 d\sigma \geq \int \varphi d\mu + \int \psi d\sigma \text{ and } \inf_{\mathbb{R}^d} \psi_0 = 0, \sup_{\mathbb{R}^d} \psi_0 \leq \|h\|_\infty, \sup_{\mathbb{R}^d} \varphi_0 \geq 0.$$

If we want to deal with smooth compactly supported functions, then the bounds get a bit worse and lead to the class (14).

In the sequel we shall take the supremum in (13) over the class Φ_h^b of admissible pairs of $C_0^\infty(\mathbb{R}^d)$ -functions such that $\|\psi\|_\infty \leq \|h\|_\infty$.

Step 4. Adjoint problem. Fix an admissible pair $(\phi, \psi) \in \Phi_h^b$. The smootheness of operators A_k^ε imply [14, Th. 3.2.1] that the following adjoint problems have solutions $g, f \in C_b^{2,1}(\mathbb{R}^d \times [0, t])$:

$$\partial_s g + \mathcal{L}_k^\varepsilon g = 0, \quad g(\cdot, t) = \phi(\cdot) \quad \text{and} \quad \partial_s f + \mathcal{L}_k^\varepsilon f = 0, \quad f(\cdot, t) = \psi(\cdot) \quad (15)$$

First, due to the maximum principle [14, Th. 3.1.1]

$$\sup_{\mathbb{R}^d \times [0, t]} |g| \leq \sup_{\mathbb{R}^d} |\phi|, \quad \sup_{\mathbb{R}^d \times [0, t]} |f| \leq \sup_{\mathbb{R}^d} |\psi|. \quad (16)$$

Let us derive bounds for $|\nabla g|$ and $|\nabla f|$. The method of doing this is inspired by the Bernstein estimates. Denote for shortness $A_k^\varepsilon := (\alpha^1, \dots, \alpha^d)$. Set $v(x, t) := |\nabla g|^2 + \kappa g^2 - t$, where κ will be chosen below. Explicit computation gives (summation over all repeated indices is assumed)

$$(\partial_s - \mathcal{L}_k^\varepsilon)v \stackrel{(15)}{=} 2\partial_{x_k} g (\partial_{x_k} q^{ij} \partial_{x_i x_j}^2 g + \partial_{x_k} \alpha^i \partial_{x_i} g) - 2q^{ij} \partial_{x_k x_i}^2 g \partial_{x_k x_j}^2 g - 2\kappa q^{ij} \partial_{x_i} g \partial_{x_j} g - 1. \quad (17)$$

Due to dissipativity, DA_k^ε defines a negative quadratic form and

$$\partial_{x_k} g \partial_{x_k} \alpha^i \partial_{x_i} g = (DA_k^\varepsilon \nabla g, \nabla g) \leq 0.$$

By virtue of this observation, (7) and the Cauchy inequality $2ab \leq ca^2 + c^{-1}b^2$ with $c = 2\nu$, (17) is dominated by

$$\omega c^{-1} |\nabla g|^2 + c \sum_{i,j} (\partial_{x_i x_j}^2 g) - 2\nu \sum_{i,j} (\partial_{x_i x_j}^2 g) - 2\nu \kappa |\nabla g|^2 - 1 = \Omega |\nabla g|^2 - 2\nu \kappa |\nabla g|^2 - 1,$$

where $\omega := 2 \max\{|\partial_{x_k} q^{ij}|\}$ and $\Omega := \omega \cdot (2\nu)^{-1}$ depend *only* on the diffusion matrix and not on the drift. Choosing $\kappa := \Omega \cdot (2\nu)^{-1}$, we get

$$(\partial_s - \mathcal{L}_k^\varepsilon)v \leq -1.$$

Therefore the maximum principle [14, Th. 3.1.1] ensures

$$\max_{\mathbb{R}^d \times [0, t]} |v| \leq \max_{\mathbb{R}^d} |v(x, 0)| \equiv \max_{\mathbb{R}^d} |\nabla \phi|^2 + \kappa \max_{\mathbb{R}^d} |\phi|^2,$$

hence

$$\sup_{\mathbb{R}^d \times [0, t]} |\nabla g(x, s)| \leq (\max_{\mathbb{R}^d} |\nabla \phi|^2 + \kappa \max_{\mathbb{R}^d} |\phi|^2)^{1/2} =: C_1. \quad (18)$$

Similarly

$$\sup_{\mathbb{R}^d \times [0, t]} |\nabla f(x, s)| \leq (\max_{\mathbb{R}^d} |\nabla \psi|^2 + \kappa \max_{\mathbb{R}^d} |\psi|^2)^{1/2} + t =: C_2. \quad (19)$$

Set $l := C_1 + C_2$.

Finally, let us prove the crucial assertion: if the pair (φ, ψ) is admissible, and f and g solve (15), then $g(x, 0) + f(y, 0) \leq h(|x - y|)$. In the case $Q \equiv I$ it was proved in [13, Th.3.1]. In the general case the proof almost repeats the case $Q = I$, but we sketch it for completeness. By approximating h from above, we can assume without lack of generality that $h \in C^1(\mathbb{R})$. Define $H(y_1, y_2) := h(|y_1 - y_2|)$ and

$$0 \leq \xi(y_1, y_2) = \xi(y_2, y_1) = \begin{cases} \frac{h(|y_1 - y_2|)}{|y_1 - y_2|} & \text{if } y_1 \neq y_2 \\ 0 & \text{if } y_1 = y_2 \end{cases}.$$

First assume that

$$\partial_s g + \mathcal{L}_k^\varepsilon g > 0, \quad \partial_s f + \mathcal{L}_k^\varepsilon f > 0.$$

Suppose that $\zeta(y_1, y_2, s) := g(y_1, s) + f(y_2, s) - H(y_1, y_2)$ attains a local maximum at (Y_1, Y_2, S) and $S < t$. Then $\partial_s \zeta(Y_1, Y_2, S) = \partial_s g(Y_1, S) + \partial_s f(Y_2, S) \leq 0$,

$$\nabla_{y_1} \zeta(Y_1, Y_2, S) = \nabla_{y_2} \zeta(Y_1, Y_2, S) = 0 \implies \nabla_{y_1} g(Y_1, S) = -\nabla_{y_2} f(Y_2, S) = \xi(Y_1, Y_2)(Y_1 - Y_2)$$

and, due to dissipativity,

$$A_k^\varepsilon(Y_1, S) \nabla_{y_1} g(Y_1, S) + A_k^\varepsilon(Y_2, S) \nabla_{y_2} f(Y_2, S) = \xi(Y_1, Y_2) \langle A_k^\varepsilon(Y_1, S) - A_k^\varepsilon(Y_2, S), Y_1 - Y_2 \rangle \leq 0.$$

Since $\zeta(Y_1 + z, Y_2 + z, S)$ as a function of z has a local maximum at $z = 0$ and Q is positive definite, $\text{trace} \bar{Q} D^2 \zeta = \text{trace} Q(Y_1, S) D^2 g + \text{trace} Q(Y_2, S) D^2 f \leq 0$, where

$$\bar{Q}(y_1, y_2, s) := \begin{pmatrix} Q(y_1) & 0 \\ 0 & Q(y_2) \end{pmatrix}.$$

Summing all up, we get

$$(\partial_s g + \mathcal{L}_k^\varepsilon g) + (\partial_s f + \mathcal{L}_k^\varepsilon f) \leq 0;$$

this contradiction means that the local maximum can be attained only at $S = t$. Now we proceed to the equality. Setting for some $\varepsilon, \delta > 0$

$$g_{\varepsilon, \delta}(y_1, s) := g(y_1, s) - \delta(t - s) - \varepsilon e^{-s} |y_1|^2, \quad f_{\varepsilon, \delta}(y_2, s) := f(y_2, s) - \delta(t - s) - \varepsilon e^{-s} |y_2|^2$$

and computing $\partial_s + \mathcal{L}_k^\varepsilon$, we come to the previous case for ε, δ small enough (since all the coefficients of the differential operator are bounded). Passing to the limit as $\varepsilon, \delta \rightarrow 0$, we come to the required assertion.

Step 5. Deriving the estimate-1. Plugging solutions of (15) into identity (4), we get

$$\begin{aligned} \int \phi d\mu_t - \int g(x, 0) d\mu_0 &= - \int_0^t \int (A_k^\varepsilon(x, s) - B_\mu) \cdot \nabla g(x, s) d\mu_s ds, \\ \int \psi d\sigma_t - \int f(x, 0) d\sigma_0 &= - \int_0^t \int (A_k^\varepsilon(x, s) - B_\sigma) \cdot \nabla f(x, s) d\sigma_s ds. \end{aligned}$$

Because of (18),

$$\int \phi d\mu_t - \int g(x, 0) d\mu_0 \leq l \int_0^t \int |A_k^\varepsilon - B_\mu| d\mu_s ds.$$

Note that

$$\begin{aligned} \int \psi d\sigma_t - \int f(x, 0) d\sigma_0 &\leq \int_0^t \int |A_k^\varepsilon - B_\mu| \cdot |\nabla f| d\sigma_s ds + \int_0^t \int |B_\mu - B_\sigma| \cdot |\nabla f| d\sigma_s ds \stackrel{(19)}{\leq} \\ &\leq l \int_0^t \int |A_k^\varepsilon - B_\mu| d\sigma_s ds + \int_0^t \int |B_\mu - B_\sigma| \cdot |\nabla f| d\sigma_s ds. \end{aligned}$$

Summing up these inequalities, we get

$$\int \phi d\mu_t + \int \psi d\sigma_t \leq \int g(x, 0) d\mu_0 + \int f(x, 0) d\sigma_0 + l \cdot R_k^\varepsilon + \int_0^t \int |B_\mu - B_\sigma| \cdot |\nabla f| d\sigma_s ds,$$

where

$$R_k^\varepsilon := \int_0^t \int |A_k^\varepsilon - B_\mu| d(\mu_s + \sigma_s) ds.$$

By virtue of step 4 we have $g(x, 0) + f(y, 0) \leq h(|x - y|)$. Thus

$$\int g(x, 0) d\mu_0 + \int f(x, 0) d\sigma_0 \leq C_h(\mu_0, \sigma_0). \quad (20)$$

So we get

$$\int \phi d\mu_t + \int \psi d\sigma_t \leq C_h(\mu_0, \sigma_0) + l \cdot R_k^\varepsilon + \int_0^t \int |B_\mu - B_\sigma| \cdot |\nabla f| d\sigma_s ds. \quad (21)$$

Step 6. Integral bound for ∇f . The last term in the right-hand side of (21) is dominated by

$$\sqrt{\int_0^t \int \nu^{-1} |B_\mu - B_\sigma|^2 d\sigma_s ds} \cdot \sqrt{\int_0^t \int |\sqrt{Q} \nabla f|^2 d\sigma_s ds}.$$

To estimate the second multiplier, i.e. $\|\sqrt{Q} \nabla f\|_{L^2(\mathbb{R}^d \times [0, T]; d\sigma_s ds)}^2$, note that f^2 is a function of the class $C_b^{2,1}(\mathbb{R}^d \times [0, t)) \cap C(\mathbb{R}^d \times [0, T])$ and it can be plugged into the identity (4) for the measure σ :

$$\begin{aligned} \int \psi^2 d\sigma_t - \int f^2(x, 0) d\sigma_0 &= \int_0^t \int (\partial_s + L_\sigma) f^2 d\sigma_s ds \\ &= \int_0^t \int 2f(\partial_s f + \text{trace}(QD^2 f) + \langle B_\sigma, \nabla f \rangle) + 2|\nabla f|^2 d\sigma_s ds \\ &= - \int_0^t \int 2f \langle A_k^\varepsilon - B_\sigma, \nabla f \rangle + 2|\nabla f|^2 d\sigma_s ds. \end{aligned}$$

Hence

$$2 \int_0^t \int |\nabla f|^2 d\sigma_s ds \leq \int \psi^2 d\sigma_t - \int f^2(x, 0) d\sigma_0 + 2 \max |f(x, t)| \int_0^t \int \nu^{-1/2} |A_k^\varepsilon - B_\sigma| \cdot |\sqrt{Q} \nabla f| d\sigma_s ds.$$

The maximum principle (16) and definition (14) imply $\max |f(x, t)| \leq \max |\psi(x)| \leq \|h\|_\infty$. Taking into account the Cauchy inequality $ab \leq 2^{-1}\gamma a^2 + (2\gamma)^{-1}b^2$ with $\gamma = \|h\|_\infty$, we come to

$$2 \int_0^t \int |\nabla f|^2 d\sigma_s ds \leq \|h\|_\infty^2 + \|h\|_\infty^2 \nu^{-1} \int_0^t \int |A_k^\varepsilon - B_\sigma|^2 d\sigma_s ds + \int_0^t \int |\sqrt{Q} \nabla f|^2 d\sigma_s ds.$$

Cancelling alike terms and recalling (21), we get

$$\int \phi d\mu_t + \int \psi d\sigma_t \leq C_h(\mu_0, \sigma_0) + l \cdot R_k^\varepsilon + \|h\|_\infty \nu^{-1/2} \cdot r_k^\varepsilon \cdot \sqrt{\int_0^t \int |B_\mu - B_\sigma|^2 d\sigma_s ds}, \quad (22)$$

where

$$r_k^\varepsilon := \sqrt{1 + \nu^{-1} \int_0^t \int |A_k^\varepsilon - B_\sigma|^2 d\sigma_s ds}.$$

Step 7. Limits as $\varepsilon \rightarrow 0$ and $k \rightarrow \infty$. Deriving the estimate-2. First of all, we recall that

$$A_k^\varepsilon(x, t) \rightarrow A_k(x, t) \text{ for a.e. } (x, t) \in \mathbb{R}^d \times [0, T],$$

and the measure $d\sigma_s ds$ and $d(\mu_s + \sigma_s)ds$ have strictly positive densities on $\mathbb{R}^d \times [0, T]$ with respect to the Lebesgue measure. Thus

$$A_k^\varepsilon(x, t) \rightarrow A_k(x, t) \quad d\sigma_s ds\text{-a.e. and } d(\mu_s + \sigma_s)ds\text{-a.e.}$$

Since for each k the mappings A_k^ε and A_k are bounded, Lebesgue's dominated convergence theorem yields

$$\begin{aligned} \int_0^t \int |A_k^\varepsilon - B_\sigma|^2 d\sigma_s ds &\rightarrow \int_0^t \int |A_k - B_\sigma|^2 d\sigma_s ds, \quad \varepsilon \rightarrow 0, \\ \int_0^t \int |A_k^\varepsilon - B_\mu| d(\mu_s + \sigma_s) ds &\rightarrow \int_0^t \int |A_k - B_\mu| d(\mu_s + \sigma_s) ds, \quad \varepsilon \rightarrow 0. \end{aligned}$$

Next, recall that

$$\lim_{k \rightarrow \infty} A_k(x, t) = B_\mu(x, t) \text{ for a.e. } (x, t) \in \mathbb{R}^d \times [0, T].$$

Similarly, taking into account (9), one can apply the Lebesgue's dominated convergence theorem and get

$$\lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} R_k^\varepsilon = 0, \quad \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} r_k^\varepsilon = \sqrt{1 + \int_0^t \int \nu^{-1} \cdot |B_\mu - B_\sigma|^2 d\sigma_s ds}.$$

Hence one can pass in (22) to limits as $\varepsilon \rightarrow 0$, then $k \rightarrow \infty$ and get

$$\begin{aligned} \int \phi d\mu_t + \int \psi d\sigma_t &\leq C_h(\mu_0, \sigma_0) \\ &\quad + \|h\|_\infty \sqrt{\int_0^t \int \nu^{-1} |B_\mu - B_\sigma|^2 d\sigma_s ds} \cdot \sqrt{1 + \nu^{-1} \int_0^t \int |B_\mu - B_\sigma|^2 d\sigma_s ds}. \end{aligned} \quad (23)$$

Passing to supremum over $(\phi, \psi) \in \Phi_h^b$ and using step 3, we obtain

$$C_h(\mu_t, \sigma_t) \leq C_h(\mu_0, \sigma_0) + \|h\|_\infty \sqrt{\int_0^t \int \nu^{-1} |B_\mu - B_\sigma|^2 d\sigma_s ds} \cdot \sqrt{1 + \nu^{-1} \int_0^t \int |B_\mu - B_\sigma|^2 d\sigma_s ds},$$

i.e. the estimate (10) with $\lambda = 0$. ■

3 Applications to nonlinear equations

In this section we focus on the applications of the obtained estimate to the study of the well-posedness of the Cauchy problem for the nonlinear FPK equations.

Given a continuous positive function α on $[0, T]$, $\tau \in (0, T]$ and a non-negative continuous function $V(x)$ on \mathbb{R}^d with $V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$, define classes of measures

$$\begin{aligned} M_{\tau, \alpha}(V) &= \{\mu = (\mu_t)_{t \in [0, \tau]} : \int V(x) d\mu_t \leq \alpha(t), t \in [0, \tau]\}, \\ M_\tau(V) &= \{\mu = (\mu_t)_{t \in [0, \tau]} : \sup_{t \in [0, \tau]} \int V(x) d\mu_t < +\infty\}. \end{aligned}$$

Throughout the section we assume that a non-degenerate $d \times d$ -matrix $Q(x, t)$ satisfying (A1) is fixed. Suppose that for each measure $\mu = \mu_t dt \in M_T(V)$ a Borel mapping

$$B(\mu, \cdot, \cdot) \equiv B(\mu) : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$$

is defined. Consider the Cauchy problem for a nonlinear FPK equation

$$\partial_t \mu_t = \text{trace}(Q(x, t) D^2 \mu_t) - \text{div}(B(\mu, x, t) \mu_t), \quad \mu_t|_{t=0} = \mu_0. \quad (24)$$

Again denote the elements of the diffusion matrix $Q(x, t)$ by $q^{ij}(x, t)$, $1 \leq i, j \leq d$ and the elements of the vector drift $B(\mu, x, t)$ by $b^j(\mu, x, t)$, $1 \leq j \leq d$. Set

$$L_\mu \phi = q^{ij}(x, t) \partial_{x_i x_j}^2 \phi + b^i(\mu, x, t) \partial_{x_i} \phi,$$

where summation over all repeated indices is taken. As earlier, we call the measure $\mu = \mu_t dt$, $t \in [0, T]$ a solution to (24), if the identity (3) holds with L_μ instead of L . Introduce the following assumptions on the drift:

(B1) The drift term B is λ -dissipative in x , i.e. for every measure $\mu \in M_T(V)$

$$\langle B(\mu, x, t) - B(\mu, y, t), x - y \rangle_{\mathbb{R}^d} \leq \lambda \|x - y\|^2 \quad (25)$$

for all $x, y \in \mathbb{R}^d$ and all $t \in [0, T]$.

(B2) for all measures μ and σ from $M_T(V)$

$$B(\mu, x, t) - \lambda x \in L^2(\mathbb{R}^d \times [0, T], d(\mu_s + \sigma_s) ds). \quad (26)$$

We start with the question of uniqueness and stability of the probability solution to (24). As earlier, we assume that some continuous non-decreasing monotone bounded cost function h with $h(0) = 0$ is fixed. Given a non-negative non-decreasing function G , denote

$$G^*(r) := \int_r^1 \frac{du}{G^2(\sqrt{u})}.$$

Corollary 3.1. *Fix some non-negative continuous function $V(x)$ on \mathbb{R}^d with $V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$ such that $V \in L^1(\mathbb{R}^d; \mu_0) \cap L^1(\mathbb{R}^d; \sigma_0)$. Assume that the coefficients of the equation (24) satisfy (A1), (B1) and (B2) with this V . Moreover, assume that for each two measures $\mu = (\mu_t)_{t \in [0, T]}$ and $(\sigma_t)_{t \in [0, T]}$ from $M_T(V)$*

$$|B(\mu, x, t) - B(\sigma, x, t)| \leq \sqrt{V(x)} G(C_h(\mu_t, \sigma_t)) \quad (27)$$

for some non-negative increasing function G such that $G^*(0) = +\infty$.

Then each two solutions $(\mu_t)_{t \in [0, T]}$ and $(\sigma_t)_{t \in [0, T]}$ of the problem (24) from the class $M_T(V)$ with initial data μ_0 and σ_0 respectively satisfy

$$C_{h, \lambda t}(\mu_t, \sigma_t) \leq \left((G^*)^{-1} \left(G^*(2(C_h(\mu_0, \sigma_0))^2) - ct \right) \right)^{1/2}$$

for all $t \in [0, T]$; here $(G^*)^{-1}$ is the inverse to G^* function, and $c > 0$ is some positive constant.

Example 3.1. Assumptions (B1), (B2) and (27) are fulfilled, for example, for drift terms of the form

$$B(\mu, x, t) = H(x) \int k(x, y) d\mu_t(y)$$

with $0 \leq H(x) \leq \sqrt{V(x)}$ and a λ -dissipative in the first variable kernel $k(\cdot, \cdot)$ such that

$$|k(x, y) - k(z, y)| \leq h(|x - y|).$$

Proof of Corollary 3.1. First of all, if μ is a solution to (24) and assumptions of Corollary 3.1 are fulfilled, then the linear FPK equation

$$\partial_t \rho_t = \text{trace}(Q(x, t) D^2 \rho_t) - \text{div}(B(\mu, x, t) \rho_t), \quad \rho_t|_{t=0} = \mu_0$$

has a solution $\rho = \mu$ and it satisfies assumptions of Theorem 2.1; similarly does σ . Hence one can apply (10) with $B_\mu(\cdot, \cdot) = B(\mu, \cdot, \cdot)$ and $B_\sigma(\cdot, \cdot) = B(\sigma, \cdot, \cdot)$.

Next, arguing as on Step 1 of the proof of Theorem 2.1, we can assume that the drift term B is dissipative. With condition (27) in hand, the estimate (10) takes the form

$$C_h(\mu_t, \sigma_t) \leq C_h(\mu_0, \sigma_0) + \|h\|_\infty \sqrt{\nu^{-1}a} \sqrt{\int_0^t G^2(C_h(\mu_s, \sigma_s)) ds} \cdot \sqrt{1 + \nu^{-1}a \int_0^t G^2(C_h(\mu_s, \sigma_s)) ds}, \quad (28)$$

where $a = \sup_{t \in [0, T]} \int V(x) d\mu_t < +\infty$ and ν is the ellipticity constant of Q . Note that $C_h(\mu_t, \sigma_t) \leq \|h\|_\infty$. Then (28) can be reduced to a weaker inequality

$$C_h(\mu_t, \sigma_t) \leq C_h(\mu_0, \sigma_0) + K \sqrt{\int_0^t G^2(C_h(\mu_s, \sigma_s)) ds} \quad (29)$$

with $K = \|h\|_\infty \sqrt{\nu^{-1}a} \cdot \sqrt{1 + \nu^{-1} \cdot T G^2(\|h\|_\infty)}$. Squaring (29) and using the inequality $(b+c)^2 \leq 2b^2 + 2c^2$, we get

$$C_h(\mu_t, \sigma_t)^2 \leq 2C_h(\mu_0, \sigma_0)^2 + 2K^2 \int_0^t G^2(C_h(\mu_s, \sigma_s)) ds.$$

If $\mu_0 = \sigma_0$, then uniqueness follows immediately due to explicit integration. In the general case the Gronwall type inequality (for example, [8, Th. 27]) implies

$$C_h(\mu_t, \sigma_t) \leq ((G^*)^{-1} (G^*(2(C_h(\mu_0, \sigma_0))^2) - 2K^2 t))^{1/2}.$$

□

A particular special case $G(u) = u$ of this latter estimate is especially interesting:

Corollary 3.2. *Let μ and σ be two solutions to (24) as in Theorem 3.1 with $G(u) = u$. Then for some $N > 0$*

$$C_{h_{\lambda t}}(\mu_t, \sigma_t) \leq \sqrt{2} C_h(\mu_0, \sigma_0) e^{Nt}.$$

In particular, if the drift is dissipative ($\lambda = 0$) or $\lambda < 0$, then

$$C_h(\mu_t, \sigma_t) \leq \sqrt{2} C_h(\mu_0, \sigma_0) e^{Nt}.$$

In some cases the estimate (10) enables to establish existence of a solution to the nonlinear equation (24). To show this, consider $h(r) = \min\{|r|^p, 1\}$ for some $p \geq 1$. Recall that in this case $C_h^{1/p}(\mu_t, \sigma_t)$ is a metric and turns the space of probability measures into a complete metric space. Moreover, convergence with respect to this metric is equivalent to weak convergence (see [6, Th. 1.1.9]).

Corollary 3.3. *Suppose there exists a function $V \in C^2(\mathbb{R}^d)$, $V \geq 1$ such that $V(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$ and there exists positive function Λ on $[0, +\infty)$ such that*

$$(L_\mu V)(x, t) \leq \Lambda(\alpha(t))(1 + V(x))$$

for each $\alpha \in C^+[0, T]$, $\tau \in [0, T]$, each $(x, t) \in \mathbb{R}^d \times [0, T]$ and each $\mu \in M_{\tau, \alpha}(V)$. Assume that the coefficients in (24) satisfy (A1), (B1) and (B2) with this function V . Assume that $B(\sigma^n) \rightarrow B(\sigma)$ in $L^2(\mathbb{R}^d \times [0, T], d\sigma_s ds)$ as $n \rightarrow \infty$ if measures $\sigma^n(dx dt) = \sigma_t^n(dx) dt$ weakly converge to a measure $\sigma(dx dt) = \sigma_t(dx) dt$ on the strip $\mathbb{R}^d \times [0, T]$.

Then for every probability measure μ^ such that $V \in L^1(\mathbb{R}^d; \mu^*)$, there exists a (local) probability solution $\mu = (\mu_t)_{t \in [0, \tau]}$ to (24) with initial condition μ^* .*

Example 3.2. Let $k(x, y)$ be a bounded function, λ -dissipative in the first variable for every $y \in \mathbb{R}^d$. Let $Q(x, t)$ be a matrix satisfying (A1). Then the Cauchy problem (24) with

$$B(\mu, x, t) = \int k(x, y) d\mu_t(y)$$

satisfies all assumptions of Theorem 3.3 with $V(x) = 1 + |x|^2$ and any probability measure ν with finite second moment.

Example 3.3. Let $V > 0$ be some C^2 -function on \mathbb{R}^d with at least linear growth. Let $g(x)$ be a λ -dissipative function such that $|g| \leq \sqrt{V}$. Let $Q(x, t)$ be a matrix satisfying (A1). Then the Cauchy problem (24) with

$$B(\mu, x, t) = g(x) \int k(y) d\mu_t(y)$$

with some non-negative continuous bounded kernel $k(y)$ satisfies all assumptions of Theorem 3.3 with any probability measure ν that integrates V .

Example 3.4. Fix $\alpha \in (0, 1)$ and a matrix Q satisfying (A1). Then the Cauchy problem (24) with

$$B(\mu, x, t) = -(|x|^{\alpha-1}x) * \mu_t$$

satisfies all assumptions of Theorem 3.3 with $V(x) = 1 + |x|^2$ and any probability measure ν with finite second moment (cf. [11, Proposition 2.1]).

Proof. As earlier, without loss of generality, the drift term is dissipative. Let $\sigma \in M_{\tau, \alpha}(V)$ for some τ, α . Consider

$$\partial_t \mu_t = \text{trace}(Q(x, t) D^2 \mu_t) - \text{div}(B(\sigma, x, t) \mu_t), \quad \mu_0 = \mu^*.$$

Note that the dissipativity of the drift ensures that it is bounded locally in (x, t) . Hence under the assumptions of the theorem there exists a unique probability solution $\mu = (\mu_t)_{t \in [0, \tau]}$ in $M_\tau(V)$ (see [12, Theorem 3.6]). Therefore the mapping $\Theta : M_{\tau, \alpha}(V) \rightarrow M_\tau(V)$

$$\mu = \Theta(\sigma) \iff \partial_t \mu_t = \text{trace}(Q(x, t) D^2 \mu_t) - \text{div}(B(\sigma, x, t) \mu_t), \quad \mu_0 = \mu^*$$

is correctly defined. It is obvious that the solutions to (24) are exactly the fixed points of the mapping Θ .

Define subclass $N_{\tau, \alpha}(V)$ of the class $M_{\tau, \alpha}(V)$ as follows:

$$N_{\tau, \alpha}(V) := \{\mu \in M_{\tau, \alpha}(V) : |\int \varphi(x) d(\mu_t - \mu_s)| \leq K(\tau, \alpha, \varphi) \cdot |t - s| \ \forall \varphi \in C_0^\infty(\mathbb{R}^d)\},$$

where

$$K(\tau, \alpha, \varphi) := \sup\{|L_\mu \varphi(x, t)|, (x, t) \in \mathbb{R}^d \times [0, \tau], \mu \in M_{\tau, \alpha}(V)\}.$$

Obviously $N_{\tau, \alpha}$ is a convex set. By virtue of [11, Corollary 4] there exist $\bar{\alpha}(t) > 0$ and $\bar{\tau} \in (0, T]$ such that $\Theta(N_{\bar{\tau}, \bar{\alpha}}(V)) \subset N_{\bar{\tau}, \bar{\alpha}}(V)$. Moreover, the class $N_{\bar{\tau}, \bar{\alpha}}(V)$ is a compact set in the topology of weak convergence of measures on the strip $\mathbb{R}^d \times [0, \tau]$ by [11, Corollary 1]. Let us check that continuity of the mapping Θ on $N_{\bar{\tau}, \bar{\alpha}}(V)$. Suppose that the sequence $\sigma^n = (\sigma_t^n) \in N_{\bar{\tau}, \bar{\alpha}}(V)$ weakly converges to $\sigma = (\sigma_t) \in N_{\bar{\tau}, \bar{\alpha}}(V)$. Set $\mu^n := \Theta(\sigma^n)$, $\mu := \Theta(\sigma)$. Due to (21) we have

$$C_h(\mu_t^n, \mu_t) \leq \sqrt{\int_0^{\bar{\tau}} \int |B(\sigma^n) - B(\sigma)|^2 d\sigma_s ds} \cdot \sqrt{1 + \int_0^{\bar{\tau}} \int |B(\sigma^n) - B(\sigma)|^2 d\sigma_s ds}.$$

Our conditions imply that the right-hand side goes to zero as $n \rightarrow \infty$. Hence μ_t^n converges to μ_t with respect to the metric $C_h^{1/p}$ and thus converges weakly. Let us show that μ^n converges to μ on the strip $\mathbb{R}^d \times [0, \bar{\tau}]$. Fix some continuous bounded function $\zeta(x, t)$. Then for each $t \in [0, \bar{\tau}]$ we have

$$\int \zeta(x, t) d\mu_t^n \rightarrow \int \zeta(x, t) d\mu_t, \quad n \rightarrow \infty.$$

Since the measures μ_t^n are probability measures and ζ is bounded, the integrals on the right-hand side are uniformly bounded and pointwise (with respect to $t \in [0, \bar{\tau}]$) convergent to $\int \zeta(x, t) d\mu_t$. Therefore Lebesgue's dominated convergence theorem ensures

$$\int_0^{\bar{\tau}} \int \zeta(x, t) d\mu_t^n dt \rightarrow \int_0^{\bar{\tau}} \int \zeta(x, t) d\mu_t dt, \quad n \rightarrow \infty.$$

By definition this means that the sequence μ^n converges weakly to μ on the strip $\mathbb{R}^d \times [0, \bar{\tau}]$.

Summarizing, we have a continuous mapping Θ on a convex compact set $N_{\bar{\tau}, \bar{\alpha}}(V)$ and maps it onto itself. The Schauder fixed-point theorem ensures that there exists a fixed point of Θ in $N_{\bar{\tau}, \bar{\alpha}}(V)$, i.e. there exists a solution $\mu = (\mu)_{t \in [0, \bar{\tau}]}$ to (24) with initial condition μ^* . \square

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